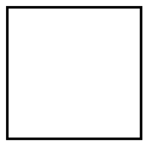
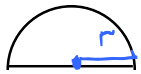
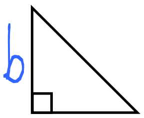
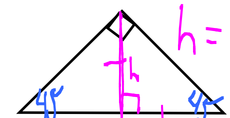
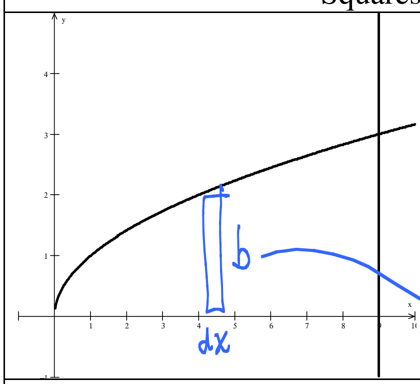
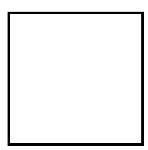
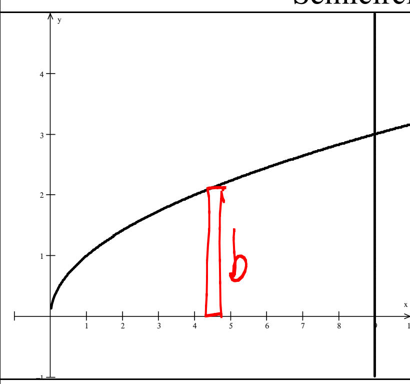
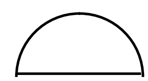
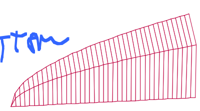
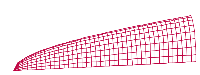
AP Calc Notes: IA – 8 Volumes with Known Cross Sections

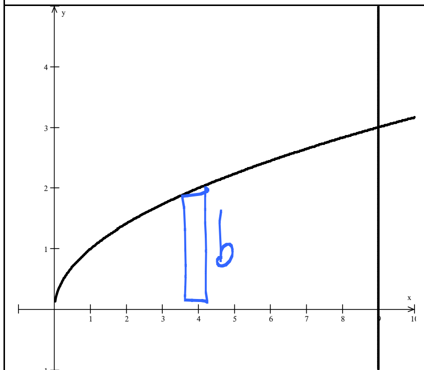
Warm-up: Write the area formulas for the following shapes

Square	Semicircle	Rectangle w/ $\boxed{h=\frac{1}{2}b}$	Isosceles right triangle w/ base as leg	Isosceles right triangle w/ base as hypotenuse
 b	 r b $r^2=\left(\frac{b}{2}\right)^2$	 $\frac{1}{2}b$ b	b  b	 $h=$ h 45° 45° b $\frac{1}{2}b$ $A=\frac{1}{2}bh$
$A=b^2$	$A=\frac{1}{2}\pi\left(\frac{b}{2}\right)^2$	$A=\frac{1}{2}b^2$	$A=\frac{1}{2}b^2$	$h=\frac{1}{2}b$ $=\frac{1}{4}b^2$ $A=\frac{1}{2}b\left(\frac{1}{2}b\right)$

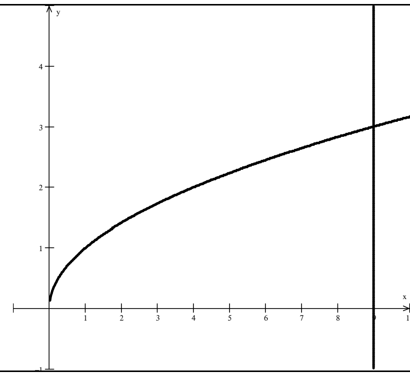
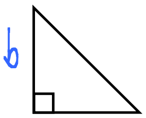
Ex: Region B is the area bounded by the x-axis, x = 9 and $y=\sqrt{x}$. Bases of cross-sections are perpendicular to the x-axis.

Squares	Semicircles
 b dx cross section  b b	 b $b=\sqrt{x}$  b
$b=y_{Top}-y_{Bottom}$ 	 $r=\frac{1}{2}b$ $=\frac{1}{2}\sqrt{x}$
$A=b^2$ $V=\int_0^9(\sqrt{x})^2\,dx$ $=\int_0^9 x\,dx=\frac{x^2}{2}\Big\vert_0^9$ $\frac{81}{2}$ No π, it is a square	$A=\frac{1}{2}\pi\left(\frac{1}{2}b\right)^2$ $V=\frac{1}{2}\pi\int_0^9\left(\frac{\sqrt{x}}{2}\right)^2dx$ $=\frac{1}{2}\pi\int_0^9\frac{1}{4}x\,dx=\frac{1}{8}\pi\frac{x^2}{2}\Big\vert_0^9$ $=\frac{81}{16}\pi$

Ex: Region B is the area bounded by the x-axis, x = 9 and $y=\sqrt{x}$. Bases of cross-sections are perpendicular to the x-axis.

<table>
<tr><th>Rectangle w/ $h=\frac{1}{2}b$</th><th>Isosceles right triangle w/ base as leg</th></tr>
<tr><td>

$A = bh$

</td><td>

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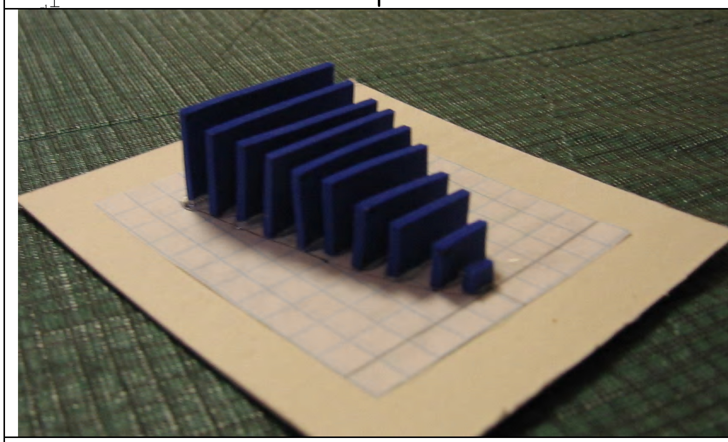
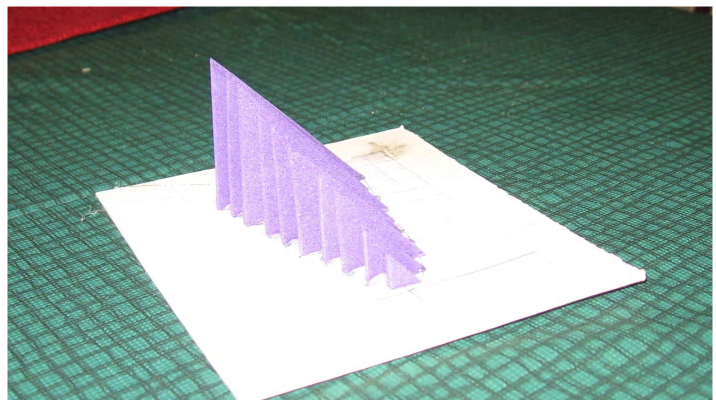
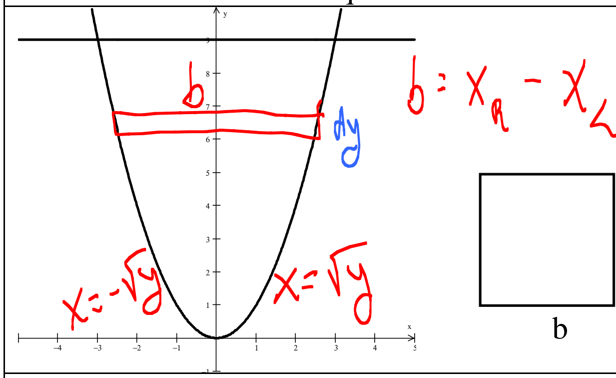
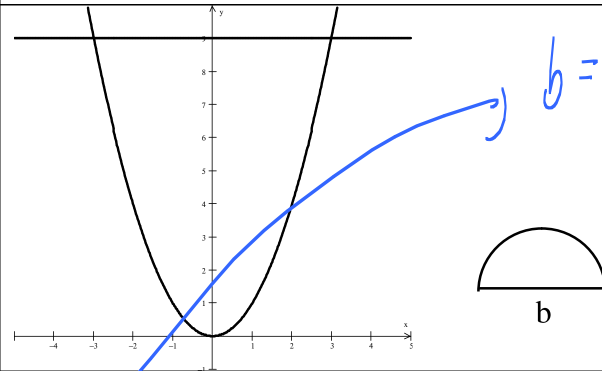
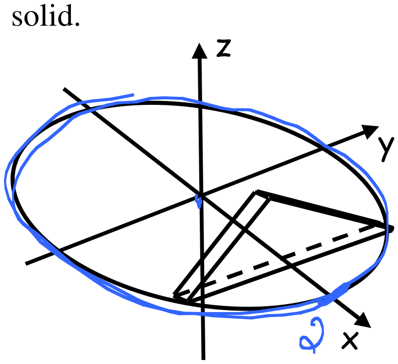
<tr><td></td><td></td></tr>
<tr><td>$A = \frac{1}{2}b^2$
$V = \frac{1}{2}\int_0^9 (\sqrt{x})^2\,dx$
$\frac{1}{2}\int_0^9 x\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)\Big|_0^9$
$= \frac{81}{4}$</td><td>$A = \frac{1}{2}b^2$
$V = \frac{1}{2}\int_0^9 (\sqrt{x})^2\,dx$
$\frac{1}{2}\int_0^9 x\,dx = \frac{1}{2}\left(\frac{x^2}{2}\Big|_0^9\right)$
$\frac{81}{4}$</td></tr>
</table>

Region A is the area bounded by $y = 9$ and $y = x^2$. Bases of cross-sections are perpendicular to the y-axis.

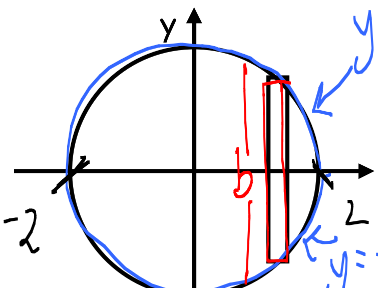
Squares	Semicircles
 $b = x_R - x_L$ $x = -\sqrt{y}$ $x = \sqrt{y}$ dy b	 $b = 2\sqrt{y}$ b
$A = b^2$ $b = \sqrt{y} - -\sqrt{y}$ $\boxed{b = 2\sqrt{y}}$ $V = \int_0^9 (2\sqrt{y})^2\,dy$ $\int_0^9 4y\,dy = \frac{4y^2}{2}\Big\vert_0^9 = 2y^2\Big\vert_0^9 = 162$	$A = \frac{1}{2}\pi\left(\frac{b}{2}\right)^2$ $V = \frac{1}{2}\pi\int_0^9 \left(\frac{\cancel{2}\sqrt{y}}{\cancel{2}}\right)^2 dy$ $\frac{1}{2}\pi\int_0^9 y\,dy = \frac{1}{2}\pi\frac{y^2}{2}\Big\vert_0^9 = \frac{81}{4}\pi$

Ex: A solid has a circular base of radius 2 in the xy-plane. Cross-sections perpendicular to the x-axis are in the shape of isosceles right triangles with their hypotenuse in the base of the solid. Find the volume of the solid.



3-D view of base region and one representative slice.

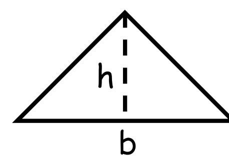
$x^2 + y^2 = 4$



$y = \sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$

View straight down on the circular base in the xy plane and on the base of the representative slice.



$A = \frac{1}{4}b^2$

View of the face of one representative slice (looking straight down the x-axis).

$$V = \int_{-2}^{2} \frac{1}{4}\left(2\sqrt{4-x^2}\right)^2 dx$$

$$V = \int_{-2}^{2} (4 - x^2)\,dx$$

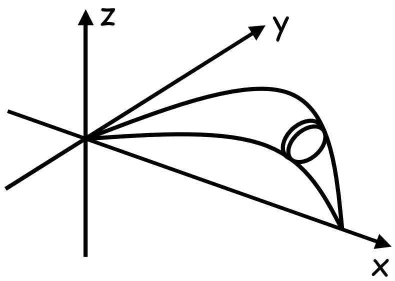
$$4x - \frac{x^3}{3}\Big\vert_{-2}^{2}$$

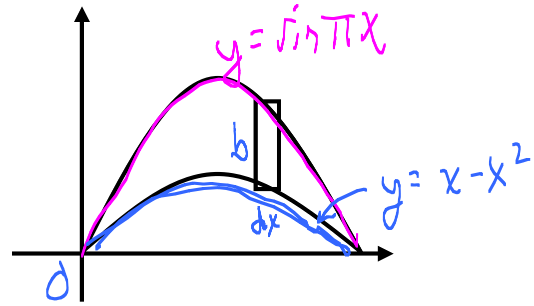
$$\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}$$

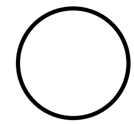
on CALC

10.667

Ex: The region R in the first quadrant is bounded by the graphs of $y = \sin \pi x$ and $y = x - x^2$. A solid is formed having circular cross-sections perpendicular to the x-axis with diameters in R. Find the volume of the solid.







$A = \pi r^2$

$b = \sin \pi x - (x - x^2)$ $\quad A = \pi\left(\frac{1}{2}b\right)^2$

$$V = \pi \int_0^1 \left(\frac{1}{2}\left(\sin \pi x - x + x^2\right)\right)^2 dx = 0.216$$

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$\sin \pi x = x - x^2$

$x = 0, \; x = 1$