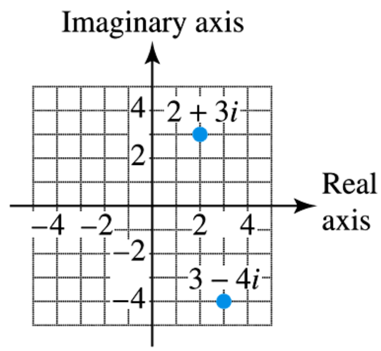
PreCalculus Class Notes PC4 Trigonometric Form and Roots of Complex Numbers

Graphing complex numbers on the Complex Plane (rectangular coordinates): the horizontal axis (x) is the real axis and the vertical axis (yi) is the imaginary axis.



Examples: Graph and label

A $2 + 0i$

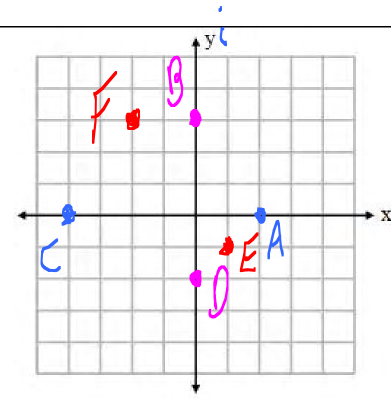
B $3i$ $\quad 0 + 3i$

C $-4 + 0i$

D $-2i$

E $1 - i$

F $-2 + 3i$



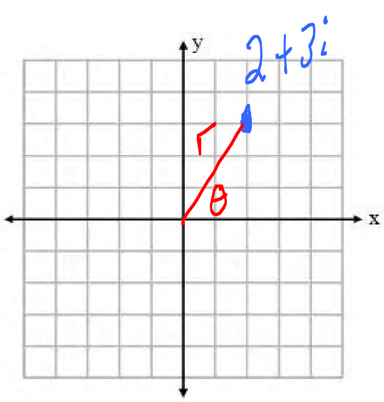
Trigonometric Form

A complex number is expressed in standard form as $a + bi$ where a is the real part and b is the imaginary part. Trig form of a complex number is similar to polar coordinates (but is not a coordinate). The expression $r(\cos\theta + i\sin\theta)$, abbreviated $r cis\theta$, is called the **trigonometric form** of the complex number $a + bi$, where $a = r\cos\theta$ and $b = r\sin\theta$. The number r is the **modulus (magnitude)** of $a + bi$, and θ is the **argument (angle)** of $a + bi$.

$r\cos\theta + r\sin\theta\, i$

Example

Find the radius and angle (nearest tenth of a degree) for the point $2 + 3i$. Write the complex number in trig form.



$r = \sqrt{2^2 + 3^2}$

$r = \sqrt{13}$

$\theta = \tan^{-1}\left(\frac{3}{2}\right)$

$\theta = 56.3°$

$r \, cis \, \theta$

$\boxed{\sqrt{13} \, cis \, 56.3°}$

$\sqrt{13}(\cos 56.3° + i \sin 56.3°)$

$\underbrace{\sqrt{13}\cos 56.3°}_{2} + i\,\underbrace{\sqrt{13}\sin 56.3°}_{3}$

$2 + i\,3$

$2 + 3i$

From standard form $a + bi$ to trig form $r(\cos\theta + i\sin\theta)$

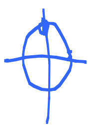
r, magnitude (modulus), distance from the origin	θ, argument, angle from the positive x-axis
$r = \lvert z \rvert = \sqrt{a^2 + b^2}$	$\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

From trig form $r(\cos\theta + i\sin\theta)$ to standard form $a + bi$

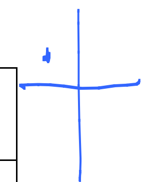
$a = r\cos\theta$	$b = r\sin\theta$

Example

Write the complex number as $a + bi$, where a and b are real numbers.



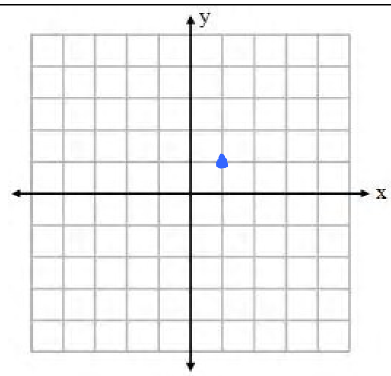
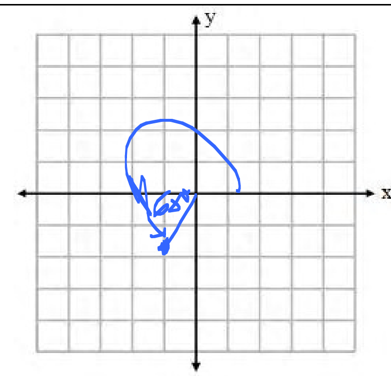
QII



$4\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$	$\sqrt{3}(\cos 150° + i\sin 150°)$
$4(0 + 1i)$ $4i$	$\sqrt{3}\left(-\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}\right)$ $-\dfrac{3}{2} + i\dfrac{\sqrt{3}}{2}$

Example

Find the trigonometric form for each complex number, where $0° \le \theta \le 360°$.

$1 + i$	$-1 - i\sqrt{3}$
	
$r = \sqrt{1^2 + 1^2} = \sqrt{2}$ $\theta = 45°$ $\sqrt{2}\ \text{cis}\ 45°$	$r = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$ $\theta = \tan^{-1}\left(\dfrac{-\sqrt{3}}{-1}\right) = 60°$ ← ref ∡ need QIII $\theta = 240°$ $2\ \text{cis}\ 240°$

Products and Quotients of Complex Numbers

Multiply two complex numbers in std form	**Divide** two complex numbers in std form
$z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i)$	$\dfrac{z_1}{z_2} = \dfrac{(a_1 + b_1 i)}{(a_2 + b_2 i)} \cdot \dfrac{(a_2 - b_2 i)}{(a_2 - b_2 i)}$
Distribute and combine	Multiply numerator and denominator by conjugate of denominator

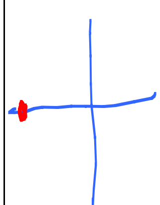
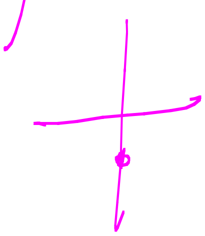
$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Multiply two complex numbers in trig form	**Divide** two complex numbers in trig form
$z_1 \cdot z_2 = r_1 \cdot r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$
Multiply r, add angles	Divide r, subtract angles

Example

Find the product and quotient of $z_1 = 4(\cos 45^\circ + i \sin 45^\circ)$ and $z_2 = 2(\cos 135^\circ + i \sin 135^\circ)$

$4 \text{ cis } 45^\circ$ $\quad$ $2 \text{ cis } 135^\circ$

Product $z_1 \cdot z_2 = r_1 \cdot r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$	Quotient $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$
$4 \cdot 2 \text{ cis}(45 + 135)$ $8 \text{ cis } 180$  -8	$\frac{4}{2} \text{ cis}(45^\circ - 135^\circ)$ $2 \text{ cis}(-90^\circ)$ $-2i$ 

Powers of complex numbers

Standard form, $(a+bi)^n$	Trig form use $\boxed{\text{De Moivre's Theorem}}$
Either distribute and combine OR Pascal's triangle, powers of terms, simplify powers of *i*, combine like terms	$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$ power on *r*, multiply angle by the value of the power

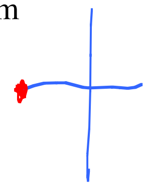
Example

Use De Moivre's theorem to evaluate $(2cis\pi)^3$

a. Write answer in trig form

$$2^3 cis(3\pi) = 8cis3\pi$$

b. Write answer in standard form



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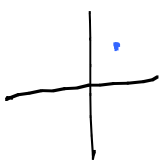
Example

Use De Moivre's theorem to evaluate $(1 + i)^8$ and express the result in standard form.

a. Write 1 + *i* in trig form



$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

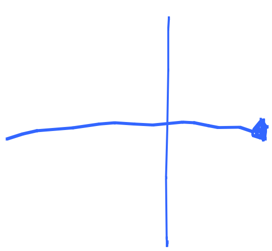
$\theta = 45^\circ$ or $\theta = \frac{\pi}{4}$

$\sqrt{2}\, cis\frac{\pi}{4}$

b. Apply De Moivre's Theorem

$$\left(\sqrt{2}\, cis\frac{\pi}{4}\right)^8 = \underbrace{\left(\sqrt{2}\right)^8}_{2^4} cis\, 8\left(\frac{\pi}{4}\right) = 16\, cis\, 2\pi$$

c. Convert complex number from trig form to standard form



16