Area, Perimeter and Volume: from rectangles to boxes

Purpose or Focus of Experience

Students will design and build a pizza box from a **single** piece of cardboard. They will analyze its surface area and volume and justify why their design is the best for their product.

Essential Question

What kinds of analysis help a packaging manufacturers design a box from a flat piece of cardboard?

Content Knowledge: Declarative, Procedural

Declarative knowledge: Students will know and/or understand:

- formulas for perimeter and area of a rectangle
- formulas for surface area and volume of a rectangular prism
- the relationship between area and perimeter of a rectangle
- the relationship between the length and width of a 2D template, the size of a cut-out square, and the surface area and volume of a 3D rectangular prism constructed from it
- which type of polynomial (linear, quadratic or cubic) model will best represent a geometric problem

Procedural knowledge: Students will be able to:

- Create scatter plots in Fathom and graph functions on this plot
- Create sliders in Fathom and use them to construct functions based on transformations
- Find linear, quadratic and cubic regression equations using a graphing calculator
- Determine the minimum and/or maximum for a function using a graphing calculator

PROCEDURE

There are 3 activities that link together the skills for this experience: Working in 2D, 2 periods, Working in 3D, 2 periods, and the Independent Project, assigned over a 10 day period. Handouts and instructions for each of these are included below. This experience involves learning new skills, such as creating sliders and graphing functions in Fathom; and learning regression on the graphing calculator. It also refines previous knowledge of the meaning of perimeter, area, surface area and volume by abstracting these concepts to new models. It refines previous knowledge of polynomials by deriving these equations and graphs from a tactile, real-life model.

Working in 2D: (2 periods)

a) Day 1: Do the Fathom project: Area and Perimeter: A study in limits available in Data in Depth: Exploring Mathematics with Fathom, Tim Erickson, Key Curriculum Press.
    i) Review scatter plots
    ii) Learn how to graph functions

iii) Learn how to create sliders and link them to formulas to transform functions

b) Day 2: Repeat the experience using a graphing calculator (Handout: Area and Perimeter: for the graphing calculator)
   i) Review scatter plots
   ii) Learn linear and quadratic regression on calculator
   iii) After students finish the graphical analysis, develop the formulas for the quadratic and linear boundaries analytically.

Area and Perimeter: for the TI-83/4 calculator

1) Use Catalog > DiagnosticOn, to show correlation coefficients later.
2) Note locations of L_1, etc on keypad as yellow characters above the numbers. Do Not TYPE L1.
3) Set up widths and lengths in lists:
   a) Stat > Edit > Edit to view lists.
   b) Clear lists if needed. (Cursor on List Name, L_1, Clear, Enter. If L_1 is missing from your lists, use Stat > Edit > Set Up Editor to reset the lists.)
   c) With cursor on List Name, set L_1 = RandInt(0,10,100) [RandInt is found in Math > PRB]

   Note: Could use rand() to include decimal lengths. Can change list names to be more descriptive.

   d) Set up L_2 similarly.
4) View a scatter plot of lengths vs widths
   a) Stat Plot > Plot 1. Turn ON Plot 1. Make sure you have scatter plot, with Xlist: L_1 and Ylist: L_2.
   b) Use ZoomStat to set the window.
5) Construct lists of the perimeters and areas
   a) With cursor on list name, set up the perimeter list, L_3 = "2(L_1+L_2)" [Note: using quotes makes the formula update if lists are changed.]
   b) Similarly, set up the area list, L_4 = "L_1*L_2"
6) View a scatter plot of the area vs the perimeter
   a) Stat Plot>Plot 1. Change the Xlist to L_3 and the Ylist to L_4.
   b) Use ZoomStat to set the window.
7) But all we want is the curve at the top. On Fathom, we deduced that these were all squares. So to show only squares, Change L_2 to equal L_1.
8) Look at the stat plot again.
9) Get the quadratic polynomial of best fit.
   a) Stat > Calc > QuadReg L_3, L_4, Y_1 > Enter [Note: if the x and y lists are L_1 and L_2, then they do not have to be typed. Y_1 is located in Vars > Y-Vars > Function > Y_1]
   b) Note equation shown.
   c) View graph.

Do homework on cut out boxes:

1. A piece of sheet metal has length L and width w (w < L). Equal size squares are to be cut from each corner and the sides bent up to make an open-topped box. Use drawing below.
   a. Express the volume of the box in terms of *x*, the length of the sides of the cutout squares.
   b. Find the domain of the function in the context of the problem.

2. Suppose the open-topped box from problem #1 was constructed from an 8 in. by 12 in. sheet of metal.
   a. What size corners should be cut out to get the largest possible volume and how much volume is obtained?
   b. What are all the possible sizes of the cutouts that would result in a box with volume of at least 60 in^3?

Answers to homework

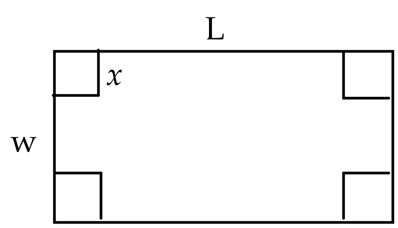
1. a. $V = x(L-2x)(w-2x)$ b. $0 < x < \frac{w}{2}$

2. a. x ≈ 1.57 in. yields a box of 67.6 in^3. b. [1 , 2.21]

Working in 3D: (2 periods)

Group project: [Need different size paper, tape and rulers.] Find the maximum volume and minimum surface area for a open-topped cut out box. Each student must do these steps individually because the groups will be mixed later.

a) Groups of four with each member in a group using the same size paper. Different groups started with different size paper. Each group member cuts out a square from each corner of the paper. Each group member should choose a different size square.

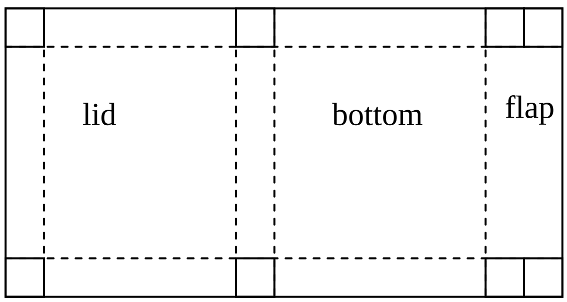


b) Create a Fathom worksheet that includes the length of the side of the cut out square, the surface area for the box and the volume for the box for all the members of the group.
c) Graph
   i) the surface area vs the side of the cutout square
   ii) the volume vs the side of the cutout square
   iii) the volume vs the surface area
d) Model these graphs with equations. Use the regression features of your calculator. Graph these as formulas on its respective Fathom graph. You decide which type of regression is best.
e) Print out the graphs with the equations.

f) Use your calculator to find the side of the square that yields the maximum volume and the minimum surface area. Is this where the volume versus surface area graph has a maximum? Annotate your graphs with this information.
g) Mix up the groups so new groups are formed with each member having data for a different size sheet of paper.
h) Generalize your findings about which type(s) of shape(s) will yield the maximum volume and/or the minimum surface area. Summarize your results.

Independent Project: (no class time allotted, project due 10 days after completion of class activities)

In groups of 2, design a pizza box. It must follow the template drawn below. The small boxes represent squares of the same size cut out so the box can be folded up to make a pizza box with a lid. You must decide the initial size of your cardboard. Support your design with graphs and equations of the surface area and volume for different sizes of cut out corners. Include all the data analysis done above, but also include information that you know about pizzas! Turn in a report with your graphs and equations. Justify why your design is the best. Include a model of your pizza box.



Note: to save paper, I made a transparency of the rubric and displayed overhead while talking with the students about the independent project.

INSTRUCTIONAL/ENVIRONMENTAL MODIFICATIONS

Students work in groups of two or four in class. Independent projects are done in groups of two, although students are allowed to complete it by themselves.

A student with limited English proficiency participated in this experience; he used a hand-held translator during class and when he wrote up his project.

To help students visualize the transition from 2D to 3D, boxes are constructed out of paper and measured with rulers. Measurements must be made in the metric or English system.

A LCP computer projector is used in the classroom, so a model of the Fathom worksheet is projected for all students to see while they are working. A TI-83 graphing calculator overhead unit is used so all students can see a model of the graphing calculator display.

For a classroom without access to computers, this project may be done solely with graphing calculators. Start the project with the graphing calculator version of Area and Perimeter. All data can be entered into the lists in the calculator. Have the students sketch their scatter plots and functions.

Students cut and pasted their Fathom graphs into a word processing document, so that the final project could be completed on any computer in the school or at home.

The four class periods can be done over four consecutive days or done in two groups of two days each separated by a week depending on the class schedule.

Students who were absent:

> Student who missed the day on Fathom had to come in during a study hall or after school and complete the Data in Depth worksheet.
>
> Students who missed the period on area and perimeter on the graphing calculator, took the worksheet home and worked through it by themselves or with help from a classmate. They had to show me their calculator with the appropriate graphs and equations to get credit for the activity.
>
> Students who missed the first day of the 3D activity were blended into a group and caught up to speed by their group members on the second day.
>
> For students who missed both days of the 3D activity, I scheduled an after school make-up session. Absent students from all three sections of class formed their own group and completed the activity.

TIME REQUIRED

Preparation time: In addition to familiarizing yourself with the lesson, you need to prepare the different size papers and collect the necessary materials. If you use different size paper than described above, you will need to make your own key. (1 – 2 class periods)

I used 4 class periods (42 minute periods) for in class activities and an extra 10 days were allotted for completion of the independent project outside of class. Students used computers during their study halls and after school to write up their project.

I need about 5 – 10 minutes per project to read the project, examine the model, verify the equations and graphs and then fill out the rubric. Other assessments take place during the class period and do not require additional time.

RESOURCES

Fathom, a software package for data analysis, available from Key Curriculum Press

Data in Depth: Exploring Mathematics with Fathom, Key Curriculum Press

A computer for each group of students (I used laptops.) Teacher's computer hooked to an LCP display unit.

TI-83 graphing calculators and overhead

Paper of various sizes: I used 8.5" x 8.5" (letter paper squared off), 8.8" x 11" (letter), 8.5" x 14" (legal), 11" x 17" (ledger)

Tape, scissors, rulers

ASSESSMENT PLAN

Formative Assessment

During the 2D activity, I observed the student groups and asked questions of individual members while they were doing the activity. At the end of each period, students were asked to summarize the main points of the lesson. Special attention was paid to:

> Period1, concluding that the rectangles on the quadratic boundary were all squares and why; why there was an empty space in the lower right and what it represented (the restriction that all lengths and widths were 10 or less.)
>
> Period 2, analytically developing the formula for the quadratic boundary and comparing it the regression formula generated by calculator.

During the 3D activity, I observed the student groups and asked questions of individual members while they were doing the activity. I also had a key of the correct formula (obtained analytically) for the surface area and volume for the boxes formed by each size paper we used. I also determined the minimum surface area and maximum volume for each size paper before the class. I used this key to quickly check that the groups were going in the right direction.

Key: Surface Area = length*width – 4*(side of square)^2

Volume = (Length – 2*side)(Width – 2*side)(side)

Initial paper size (all measurements in inches)	Cut-out square side for minimum surface area (limited domain)	Minimum surface area (nearest hundredth)	Cut-out square side for maximum volume (nearest hundredth)	Maximum Volume (nearest hundredth)
11 x 17	X = 5.5	SA = 66	X = 2.18	V = 182.97
8.5 x 14	X = 4.25	SA = 46.75	X = 1.71	V = 91.91
8.5 x 11	X = 4.25	SA = 21.25	X = 1.59	V = 66.15
8.5 x 8.5	X = 4.25	SA = 0	X = 1.42	V = 45.49

During the end of the period summary, I encouraged the students to focus on the units involved and how these could help determine the best polynomial model for each graph. In particular, surface area is measured in square inches so a quadratic is best and volume is measured in cubic inches so a cubic is best. Challenge: what type of model should be used for the volume versus surface area graph?

I counted completion of each of these activities as assignment grades (for a total of 4 grades.)

Summative Assessment

The following rubric was used to score the students' independent project. I counted the project as a test grade.

Name_____________________________ Score_______________

Polynomial Independent Project

0	-	√-	√	√+	+
Not done	Below standard on a major point or a major mistake	Below standard on a minor point or a small mistake	Meets standard, all areas correctly performed	Above standard on a minor point or in small way	Above standard on a major point or in an impressive way
-10	-6	-3	0	+3	+6

Criteria	0	-	√-	√	√+	+
Box design: initial size described, template followed						
Surface Area: equation, graph, description for your box (points - double weight)						
Volume: equation, graph, description for your box (points - double weight)						
Justification: why your box is best						
Model: quality						
Report: clear, correct spelling, organized, appearance						

Total points are added/subtracted from 100 points. Max score: 105 points

STUDENT WORK

The students' pizza boxes were displayed on a bulletin board in the classroom.

REFLECTION

Why I developed this experience: This activity grew out of my experience with calculus students. A box made from a flat template is a common starting point for calculus problems. In the past, my calculus students have had a difficult time visualizing how to form a 3D object from a 2D template. They also found it difficult to determine how to find surface area and volume unless a projective diagram was drawn for them. Clearly, students need to be exposed to the idea of folding a 2D template into a 3D object before they get to calculus. I used this box problem as a motivator to analyze all the polynomial models you could get from rectangles and rectangular prisms.

What I learned: The students needed the paper manipulative when they first started folding boxes; but, after they folded about two boxes, they would frequently do the rest of the work on paper and in their heads. The idea that the side of the cut-out square could not be any larger than half the smallest side of the paper was elusive for many students. At this point, some had to go back to cutting out a box to see that it was impossible. This natural restriction on the domain of the function reinforced previous class work on functions and I was surprised at how difficult the students found it.

Changes for next time: I would assign the following as homework the day before the 3D class activity: Have each student cut out a given template at home and determine the surface area and volume for its box. Conjecture how the surface area and volume depend on the size of the cut out square. Then students would start the class activity from step b.

During the 3D class activity, I would have the students derive the surface area and volume formulas analytically as a last step. I would delete the volume versus surface area graph and model. It went beyond the parameters of the activity and caused more confusion than it was worth.